MOTION OF A VISCOPLASTIC BODY BETWEEN TWO COAXIAL CONES

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The stream function is obtained in the form of an expansion in powers of $1/r$. It is shown that the nature of the velocity distribution is determined by the basic function of the stream function. The results obtained are compared with experimental data.

Materials such as processed peat, plastic clay, petroleum products, dough, paint, etc. are known to possess viscoplastic properties.

These properties have been the subject of a number of experimental studies [7–11], etc.

Theoretical studies in rheology lead to the necessity of integrating nonlinear differential equations of higher orders.

The theory of motion of viscoplastic bodies has been the subject of research by M. P. Volarovich, A. M. Gutkin, A. Kh. Kim, A. Kh. Mirzadzhanzade, P. K. Shchipanov [1–5], and other authors.

This article considers the problem of the motion of a viscoplastic body between two coaxial cones. A general method of solving the problem of motion of a viscoplastic body in a cone was given by Kim in [3]. The analogous problem for a viscous liquid was solved by Slezkin [6]. As Lozovskii [10] has shown, the results of this solution are applicable to viscoplastic bodies such as peat.

The equations of motion of a viscoplastic body can be obtained as a result of the joint solution of the Hencky-Il'yushin rheological equation of state

$$\Pi_0 = 2(\eta + \tau_0/h)\,\dot{\Phi}_0$$

and the Cauchy equilibrium equation

$$\operatorname{div}\Pi = \rho_0\bar{a}$$

(the body forces are relatively small and may be neglected),

$$2\left(\eta + \frac{\tau_0}{h}\right)\operatorname{div}\dot{\Phi}_0 - 2\frac{\tau_0}{h^2}\operatorname{grad}h\cdot\dot{\Phi}_0 - \operatorname{grad}p = \rho_0\bar{a}. \quad (1)$$

The problem may conveniently be solved in conical coordinates r, Θ, α. The motion is assumed to be stationary and axisymmetric. Then the following remarks may be made regarding the projections of the velocity: $v_r = v_r(r,\Theta)$; $v_\Theta = v_\Theta(r,\Theta)$; $v_\alpha = 0$.

Solving jointly the equations obtained as a result of projecting Eq. (1) onto the r and Θ axes and introducing the notation

$$\lambda = 2\left(\eta + \frac{\tau_0}{h}\right)\frac{\partial v_r}{\partial r},$$

$$\nu = 2\left(\eta + \frac{\tau_0}{h}\right)\left(\frac{1}{r}\frac{\partial v_\Theta}{\partial\Theta} + \frac{v_r}{r}\right),$$

$$\tau = \left(\eta + \frac{\tau_0}{h}\right)\left(\frac{1}{r}\frac{\partial v_r}{\partial\Theta} + \frac{\partial v_\Theta}{\partial r} - \frac{v_\Theta}{\tau}\right),$$

we obtain the equation

$$r\frac{\partial^2(\lambda-\nu)}{\partial r\,\partial\Theta} - 4r\frac{\partial\tau}{\partial r} - r^2\frac{\partial^2\tau}{\partial r^2} + \frac{\partial^2\tau}{\partial\Theta^2} - r\,\mathrm{ctg}\,\Theta\frac{\partial(2\nu+\lambda)}{\partial r} + \frac{\partial(3\lambda+\tau\,\mathrm{ctg}\,\Theta)}{\partial\Theta} = \rho_0 r\left[\frac{\partial\left(\frac{dv_r}{dt}\right)}{\partial\Theta} - \frac{2v_\Theta}{r}\frac{\partial v_\Theta}{\partial\Theta} - \frac{\partial\left(r\frac{dv_\Theta}{dt} + v_r v_\Theta\right)}{\partial r}\right]. \quad (2)$$

This is a fourth-order nonlinear partial differential equation. In the problem considered it can be reduced to a third-order linear ordinary differential equation.

In order to find the unknown functions v_r and v_Θ we add to Eq. (2) the strain continuity equation $\operatorname{div} v = 0$. For the given problem it takes the form $(1/r)\cdot(\partial/\partial r)(r^2 v_r) + (1/\sin\Theta)(\partial/\partial\Theta)(v_\Theta\sin\Theta) = 0$.

We will perform the calculations in the coordinates r, $s = \sin\Theta$, α. Moreover, we introduce the stream function $\psi(r, s)$, representing it in the form $\psi(r,s) = \sum_{i=0}^{\infty}\frac{\varphi_i(s)}{r^i}$, where $\varphi_0(s)$ is the basic function of the stream function, $\varphi_i(s)$ are the correcting functions of the stream function, $i \neq 0$. The velocity components are expressed in terms of the stream function as follows:

$$v_r = \frac{1}{r^2\sin\Theta}\cdot\frac{\partial\psi}{\partial\Theta}, \quad v_\Theta = -\frac{1}{r\sin\Theta}\frac{\partial\psi}{\partial r}.$$

We will formulate the problem of finding the basic functions $\varphi_0(s)$. In this case

$$v_r = \frac{\sqrt{1-s^2}}{r^2 s}\frac{d\varphi_0}{ds} = \frac{\varphi(s)}{r^2}, \quad v_\Theta = 0.$$

If we neglect terms of the order of $1/r^3$, Eq. (2) takes the form

$$\frac{\partial^2\tau}{\partial\Theta^2} + \frac{\partial(3\lambda + \tau\,\mathrm{ctg}\,\Theta)}{\partial\Theta} = 0.$$

The first integral of this equation is

$$\frac{\partial\tau}{\partial\Theta} + 3\lambda + \tau\,\mathrm{ctg}\,\Theta = C. \quad (3)$$

We express the left side of Eq. (3) in terms of $\varphi(s)$,

$$\left\{\frac{\varphi''\sqrt{1-s^2} - s\varphi'/\sqrt{1-s^2}}{\sqrt{12\varphi^2 + (\varphi')^2(1-s^2)}} - \frac{(\varphi')^2\sqrt{1-s^2}\,[12\varphi + \varphi''(1-s^2) - s\varphi']}{[12\varphi^2 + (\varphi')^2(1-s^2)]^{3/2}}\right\}\sqrt{1-s^2} -$$

$$-\frac{12\varphi}{\sqrt{12\varphi^2+(\varphi')^2(1-s^2)}}+$$

$$+\frac{\varphi'\sqrt{1-s^2}}{\sqrt{12\varphi^2+(\varphi')^2(1-s^2)}}\frac{\sqrt{1-s^2}}{s}=C. \qquad (4)$$

We introduce the functions

$$\Phi=\varphi'\sqrt{1-s^2}\,/\sqrt{12\varphi^2+(\varphi')^2(1-s^2)},$$
$$U=-12\varphi/\sqrt{12\varphi^2+(\varphi')^2(1-s^2)}.$$

Equation (4) then takes the form

$$\sqrt{1-s^2}\,\frac{d\Phi}{ds}+U+\frac{\sqrt{1-s^2}}{s}\,\Phi=C,$$

or

$$\frac{d(s\Phi)}{ds}=\frac{(C-U)s}{\sqrt{1-s^2}},$$

whence

$$s\Phi=-C\sqrt{1-s^2}+$$
$$+12\int\frac{\varphi s\,ds}{\sqrt{1-s^2}\sqrt{12\varphi^2+(\varphi')^2(1-s^2)}}+C_1. \qquad (5)$$

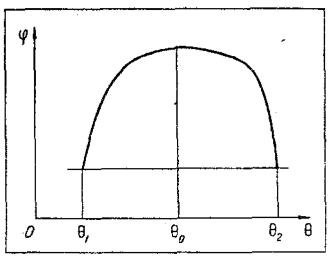

Fig. 1. Relation between the variables Θ and φ.

When the quantity $\Theta_2-\Theta_1$ is small, the integral in the right side of (5) can be neglected,

$$s\Phi=-C\sqrt{1-s^2}+C_1,$$

or

$$\frac{s\varphi'\sqrt{1-s^2}}{\sqrt{12\varphi^2+(\varphi')^2(1-s^2)}}=C_1-C\sqrt{1-s^2}. \qquad (6)$$

We denote $\varphi'/\varphi=q$; then Eq. (6) takes the form

$$sq\sqrt{1-s^2}\,/\sqrt{12+q^2(1-s^2)}=C_1-C\sqrt{1-s^2}.$$

We solve this equation for q,

$$q=\pm\sqrt{12}\,(C_1-C\sqrt{1-s^2})/$$
$$/\sqrt{(1-s^2)\left[s^2-(C_1-C\sqrt{1-s^2})^2\right]}.$$

The relation between φ and Θ is shown graphically in Fig. 1.

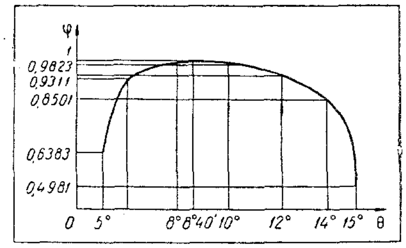

Fig. 2. Theoretical velocity distribution.

On the interval $\Theta_1<\Theta<\Theta_0$, $q>0$, on the interval $\Theta_0<\Theta<\Theta_2$, $q>0$ (Θ_0 is the point at which the function $\varphi(\Theta)$ reaches its maximum value).

According to the experiments of a number of investigators, for example, N. V. Lozovskii [10], the velocity increases very rapidly close to a solid wall. Accordingly, we may assume that $\varphi'(\Theta_1)\to\infty$ and $\varphi'(\Theta_2)\to-\infty$. Since $\varphi(\Theta_1)\neq 0$ and $\varphi(\Theta_2)\neq 0$, $q(\Theta_1)\to+\infty$ and $q(\Theta_2)\to-\infty$.

Taking into account the above, for the interval $\Theta_1<\Theta<\Theta_0$ we have

$$q=\sqrt{12}\,(C_1-C\sqrt{1-s^2})/$$
$$/\sqrt{(1-s^2)\left[s^2-(C_1-C\sqrt{1-s^2})^2\right]}. \qquad (7)$$

Since $q(s_1)\to\infty$, $C_1-C\sqrt{1-s_1^2}=s_1$. In view of the fact that $q(s_2)\to-\infty$, we obtain $C_1-C\sqrt{1-s_2^2}=-s_2$. From the system of equations

$$C_1-C\sqrt{1-s_1^2}=s_1,$$
$$C_1-C\sqrt{1-s_2^2}=-s_2$$

we find the constants of integration C and C_1:

$$C=(s_1+s_2)/(\sqrt{1-s_2^2}-\sqrt{1-s_1^2}),$$
$$C_1=s_1+\frac{s_1+s_2}{\sqrt{1-s_2^2}-\sqrt{1-s_1^2}}\sqrt{1-s_1^2}.$$

Since $q(\Theta_0)=0$, we have one more equation,

$$C_1-C\sqrt{1-s_0^2}=0,$$

from which we find the $s=s_0$ at which the function $\varphi(s)$ reaches a maximum,

$$s_0=\pm\sqrt{1-C_1^2/C^2}. \qquad (8)$$

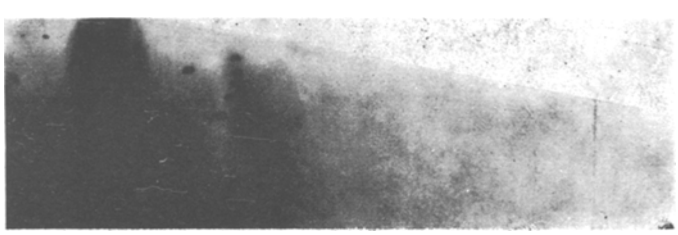

Fig. 3. Motion of lead tracers characterizing the motion of viscoplastic particles.

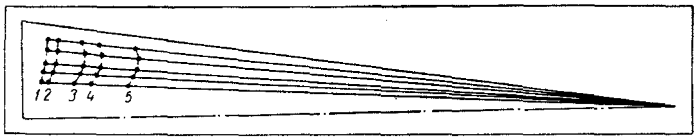

Fig. 4. Positions of tracers characterizing the velocity distribution at different pressures: 1) p = 0, 2) 0.495 atm, 3) 0.7, 4) 0.8, 5) 0.995.

We now integrate Eq. (7),

$$\frac{\varphi'}{\varphi}=\frac{\sqrt{12}\,(C_1-C\sqrt{1-s^2})}{\sqrt{(1-s^2)\left[s^2-(C_1-C\sqrt{1-s^2})^2\right]}},$$

$$\ln\varphi=\int\frac{\sqrt{12}\,(C_1-C\sqrt{1-s^2})}{\sqrt{(1-s^2)\left[s^2-(C_1-C\sqrt{1-s^2})^2\right]}}\,ds+C_2.$$

We find the constant of integration C_2 by setting $\varphi(s_0) = 1$. Then $C_2 = 0$. Thus for the function $\varphi(s)$ find the expression

$$\varphi(s)=\exp\left(\sqrt{12}\int\limits_{s_0}^{s}\frac{C_1-C\sqrt{1-s^2}}{\sqrt{(1-s^2)\left[s^2-(C_1-C\sqrt{1-s^2})^2\right]}}\,ds\right).\quad(9)$$

The integral can be evaluated numerically. Then

$$v_r=a\varphi/r^2,$$

where

$$a=\pm Q/2\pi\,[\varphi(s_2)-\varphi(s_1)].$$

Using Eq. (9), we will consider an example taking $\Theta_1 = 5°$ and $\Theta_2 = 15°$.

The graph of the velocity function obtained is shown in Fig. 2.

Using Eq. (8) we find that for the given specific case $\Theta_0 = 8°\,40'$.

Thus, we have found the basic function of the stream function $\varphi_0(s)$. In order to determine the correcting function $\varphi_1(s)$ it is necessary to represent the stream function in the form

$$\psi(r,\ s)=\varphi_0(s)+\varphi_1(s)/r.$$

Performing calculations similar to those involved in constructing differential equation (2), from which we obtained the basic function, we can obtain an equation in which the unknown function is the function $\varphi_1(s)$.

But it is possible to show that this will be a fourth-order homogeneous linear equation with variable coefficients and zero initial conditions; therefore $\varphi_1(s) \equiv 0$. Quite analogously, $\varphi_2(s) \equiv 0$. For this reason the stream function can be written in the form

$$\psi(r,\ s)=\varphi_0(s)+\varphi_3(s)/r^3.$$

In order to determine the correcting function $\varphi_3(s)$ we can use the equation

$$\frac{1}{r}\frac{\partial^2\nu}{\partial\Theta^2}-r\frac{\partial^2\nu}{\partial r^2}-2\frac{\partial^2}{\partial r\partial\Theta}[(\varepsilon+2\tau)\operatorname{ctg}\Theta]+\frac{1}{r}\frac{\partial}{\partial\Theta}[\operatorname{ctg}\Theta(\nu-6\tau-6\varepsilon)]+2\frac{\partial}{\partial\tau}[(\tau+2\varepsilon)\operatorname{ctg}^2\Theta-2\nu]=$$

$$=-2\frac{\rho_0}{r^5}\frac{\partial}{\partial\Theta}\left[\operatorname{ctg}^2\Theta\,(\varphi_0'(s))^2\right],\quad(10)$$

where

$$\tau=(\eta/r^3+\tau_0/hr^3)\left[\varphi_0'+(4s\varphi_3'-3\varphi_3)/r^3s\right],$$

$$\nu=\left(\frac{\eta}{r^3}+\frac{\tau_0}{hr^3}\right)\left(\bar{s}^2s\varphi_0''-\varphi_0'+\frac{\bar{s}^2s\varphi_3''-\varphi_3'-18s\varphi_3}{r^3}\right),$$

$$\varepsilon=(\eta/r^3+\tau_0/hr^3)\left[\varphi_0'+(\varphi_3's+3\varphi_3)/sr^3\right],$$

$$\bar{s}=\operatorname{ctg}\Theta=\sqrt{1-s^2}/s.$$

With respect to the function $\varphi_3(s)$ Eq. (10) is a fourth-order linear inhomogeneous equation with variable coefficients. Neglecting terms of the order of $1/r^4$, it can be represented in the form

$$\sum_{i=0}^{4}A_i\varphi_3^{(i)}(s)=A,$$

where A and A_i are functions depending on s, $\varphi_0(s)$, τ, η.

In solving other problems similar to that considered, Kim [3] showed that the correcting functions have very little effect on the velocity and that its nature is determined by the basic function $\varphi_0(s)$. This is also confirmed by the experimental data we have obtained. A number of experiments were performed to investigate the motion of a peat mass. For this purpose we constructed two coaxial cones for which $\Theta_1 = 5°$ and $\Theta_2 = 15°$. Lead tracers were placed in the peat mass which moved under pressure, the positions of the tracers at different pressures being photographed with an X-ray apparatus.

The photos obtained (Fig. 3) show that the streamlines are straight lines drawn from the vertex of the cone (Fig. 4). This confirms that the stream function is determined by the basic function and is relatively unaffected by the correcting function.

Moreover, the experiment confirmed the correctness of the velocity distribution curve obtained theoretically. Figure 5 shows a graph of the velocity function for peat with a moisture content of 86% at different pressures and flow rates and, for comparison, the velocity curve obtained theoretically from Eq. (9). These curves approach each other in the middle but diverge considerably at the edges, at the walls of the cones. The experimental curve shows that slip occurs near the walls when a viscoplastic body moves under the boundary conditions considered.

Apart from the velocity distribution, it is also of interest to determine one other unknown in Eq. (11), namely, the pressure. For this purpose we can use the equations obtained as a result of projecting Eq. (1) onto the coordinate axes. After the velocity has

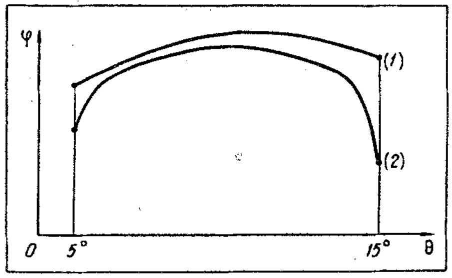

Fig. 5. Velocity distribution curves obtained experimentally (1) and theoretically (2).

been determined these equations can be written in the form

$$\frac{\partial p}{\partial r} = f_1(r, \Theta), \qquad \frac{\partial p}{\partial \alpha} = 0, \tag{11}$$

where f_1 and f_2 are known functions. The solution of system (11) has been investigated in the literature [3].

NOTATION

Π_0 is the stress tensor deviator; η is the plastic viscosity; τ_0 is the limiting shear stress; h is the strain rate intensity; $\dot{\Phi}_0$ is the strain rate tensor deviator; Π is the stress tensor; ρ_0 is the density; a is the acceleration, p is the pressure creating motion; Θ is the angle between axis of cone and radius r drawn from vertex of cone to given point; $2\Theta_1$ and $2\Theta_2$ are the plane angles at cone vertices; Q is the flow rate.

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